CHAPTER 32

Mathematical Expertise

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Competence in mathematics is a basic requirement for effective citizenship in a modern numerate society (Cockcroft, 1982). Poor numeracy skills are known to be a serious handicap for paid employment in the US (Rivera-Batiz, 1992) and the UK (Bynner & Parsons, 1997). Indeed, the UK Basic Skills Agency has published a report suggesting that numeracy is more important even than literacy in terms of career prospects in the UK (Bynner & Parsons, 1997). And the trend is toward an even greater emphasis on numeracy: recent research for the British Science, Technology and Mathematics Council shows that "mathematical skills in the workplace are changing, with increasing numbers of people engaged in mathematics-related work, and with such work involving increasingly sophisticated mathematical activities" (Hoyles, Wolf, Molyneux-Hodgson, & Kent, 2002).

The level of competence routinely demanded in numerate cultures today would have been considered quite exceptional 200 years ago. How then does one distinguish today's expert from the normally competent school-leaver who can handle numbers of arbitrary size, fractions and decimals, logarithms, equations with unknowns and negative roots, and some differentiation and integration? One could arbitrarily take the top *n*% of a standard test (like the SAT-M), but what should *n* be? Francis Galton, in *Hereditary Genius*, used obituaries from *The Times* of London and a biographical dictionary, *Men of our Time*, as the criteria of "eminence." This gave him an estimated proportion of 0.025% of the population. Really exceptional individuals, his class G, were about one-twentieth of these. He even designated a class X of people who were fewer than one in a million (Galton, 1979).

However, not every expert that we would wish to consider has taken the SAT-M, or been the subject of a *Times* obituary. The important criterion is that the candidate expert demonstrates "reproducible superior performance," preferably under something like laboratory conditions (Ericsson & Charness, 1994).

The focus of this chapter will be to attempt to identify the cognitive capacities, the disposition, and the training that

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equip someone to demonstrate "reproducible superior performance," particularly in calculation, since this is the only area of expertise in which there is much evidence.

There have been three extensive early reviews of exceptional mathematical abilities. The first was E. W. Scripture, a psychologist at Leipzig (Scripture, 1891). He reviewed in some detail the lives of 12 "arithmetical prodigies," including one mathematician of note, Carl Friedrich Gauss. From a "psychological analysis" of these lives, he identified five characteristics that seemed to distinguish the prodigies: the accuracy and "rapidity" of memory, "arithmetical association" (knowing lots of arithmetical facts and procedures), inclination, mathematical precocity, and "imagination" (visual imagery).

The second review, like Scripture's, was published in the *American Journal of Psychology*, and was by Frank D. Mitchell of "The Psychological Seminary of Cornell University" (Mitchell, 1907). Also like Scripture, he reviews the lives of prodigies, the same ones as Scripture plus a further case, the author. These are summarized in a table that lists the heredity, development, education, mental calculation, and memory of each prodigy. Mitchell takes the view that prodigious abilities grow out of counting, a verbal skill where the numbers are recited out loud initially, and internally thereafter. This has an implication for the "memory type" used by prodigies. In his own case and that of most of the prodigies he has analyzed, the memory is of the auditory type, reflecting this early experience since most of us begin to learn numbers through counting, which involves spoken words. Only between two and four of the prodigies he examined have memory of the "visual type."

The third review, by Alfred Binet (Binet, 1894), included reaction-time tests of two theatrical calculators of his day.

More recently, Barlow (1952) has written about mathematical prodigies in the context of other kinds of prodigy (Barlow, 1952). And there is an excellent reconsideration of the calculators studied by Scripture, Binet, and Mitchell, along with useful data on more recent calculators by Steven B. Smith in *The Great Mental Calculators: The Psychology, Methods, and Lives of the Calculating Prodigies* (Smith, 1983). I shall be drawing heavily on this volume along with Scripture, Mitchell, and Binet for biographical details of calculating prodigies. Finally, one of the leading modern investigators has published a review of expert calculators (Pesenti, 2005).

Calculators have attracted also the attention of experimental psychologists usually focusing on a single case. The Polish calculator Salo Finkelstein was studied by Weinland and Schlauch (1937), who used sophisticated analyses of careful timing data from various mathematical tasks, but with no control subjects! The British mathematician Alexander Aitken was studied by Hunter (1962), and a more recent case, Rüdiger Gamm, by Pesenti and colleagues (Pesenti et al., 2001). Only Binet (1894) seems to have explored the general phenomenon, carrying out experiments on two professional theatrical calculators and comparing their results with other groups of practiced and unpracticed calculators. He took considerable pains to find the optimal way of timing the stimulus presentation and the response, in an age when there were no voice-activated relays. However, he was less careful in designing the experiments so that all the subjects received the same stimuli under the same conditions.

What is striking about all these students of exceptional mathematical abilities is their conviction that there is nothing special about mathematics as a cognitive domain. This may be contrasted with Gardner's popular theory of "multiple intelligences," one of which is "logico-mathematical" (Gardner, 1983).

The research on mathematical, especially numerical, abilities in general strongly suggests the existence of a domain-specific capacity. There are two main arguments in support of this.

First, specialized brain areas, especially the left angular gyrus and the intraparietal sulci, are active when mathematical activities are taking place (Donlan, 2003; Gruber, Indefrey, Steinmetz, & Kleinschmidt, 2001; Pesenti, Thioux, Seron, & De Volder, 2000). When these areas are damaged, selective impairment of calculation frequently occurs (Cipolotti & van Harskamp, 2001). Notice that the specialized number areas are distinct from the areas active in reasoning, which are in the prefrontal cortex (e.g., Goel & Dolan, 2004).

Second, there is evidence for an innate basis to this specialization. One strand of evidence supports the claim for numerical capacity in infants, even in the first few months of life, when neither language nor frontal-lobe functions such as reasoning have developed. They are able to respond discriminatively on the basis of numerosity (Antell & Keating, 1983; Starkey & Cooper, 1980) and can even mentally manipulate numerosities at six months by working out what would happen when an object is added or subtracted from an array (Wynn, 1992, 2000, 2002; Wynn, Bloom, & Chiang, 2002). These infant capacities are similar to ones observed in monkeys (Hauser, MacNeilage, & Ware, 1996), which suggests ancestral versions that may have evolved because the ability to recognize the numerosity parameter in the environment offers advantages in foraging, mating (Edwards, Alder, & Rose, 2002), and also in conflict with conspecifics (McComb, Packer, & Pusey, 1994). (See Butterworth, 1999, for a review.) Moreover, developmental disorder can lead to selective deficits in the acquisition of even these simple numerical concepts when intelligence, memory, and language are all at normal levels (Landerl, Bevan, & Butterworth, 2004).

However, even if it is accepted that we humans have inherited a specialized capacity for representing numerosities, it does not follow that this is normally distributed, such that some people have it to a greater degree than others, like height or IQ. It may be more like color vision – either it is normal, or it is defective in one of a small number of ways. In the same way that ability as a colorist may require normal color vision, the range of colorist abilities is not determined by better or worse color discrimination, so it is possible that calculating abilities require a normal numerical "starter kit," but the abstract and complex skills that make expert mathematicians and calculators is built by other means on this basis. Smith (1983) compares it to juggling. "Any sufficiently diligent nonhandicapped person can learn to juggle, but the skill is actually acquired only by a handful of highly motivated individuals" (p. 6). Calculating prodigies themselves often say that their abilities come from their interest in numbers rather than from some special gift.

So our central question in this chapter is whether mathematical expertise, and in particular calculation expertise, depends on high cognitive abilities in non-numerical domains, such as reasoning and memory, or whether it can exist as a domain-specific achievement.

What Makes for Mathematical Expertise?

Francis Galton was quite clear that any kind of eminence depended on "natural ability," which was by-and-large inherited. "By natural ability, I mean those qualities of intellect and disposition, which urge and qualify a man to perform acts that lead to reputation. I do not mean capacity without zeal, nor zeal without capacity, nor even a combination of both of them, without an adequate power of doing a great deal of very laborious work."

Having compared divines, wrestlers, men of science, painters, poets, and others, he found that eminent people tended to have eminent parents and to produce eminent offspring. Of course, we are talking mostly about eminent men and their fathers, since it was difficult for women to achieve eminence. Overall, 31% of eminent men had an eminent parent. Some 26% of scientific men had scientific fathers, but 60% of scientifically eminent fathers have eminent sons (vs. 41% average). Galton speculates "descendants [are] taught . . . not to waste [their] powers on profitless speculation" (p. 320).

Although inherited characteristics are held to be the key, "it may be well to add a few supplementary remarks on the small effects of a good education on a mind of the highest order. A youth of abilities G, and

X, is almost independent of ordinary school education" (p. 43). He gives as an example D'Alembert, who was a "foundling . . . put out to nurse as a pauper baby to the wife of a poor glazier. The child's indomitable tendency to the higher studies, could not be repressed by his foster mother's ridicule and dissuasion, nor by the taunts of his schoolfellows, nor by the discouragements of his schoomaster, who was incapable of appreciating him, nor even by the reiterated deep disappointment of finding that his ideas, which he knew to be original, were not novel, but long previously discovered by others" (pp. 43–44). He records many other eminent men with a comparably unpromising history.

He pays particular attention to "Wranglers," students of mathematics at Cambridge University. Not only do these men have to pass the entrance examinations to Cambridge (not as difficult then as now, I believe), but they were ranked from the highest (Senior) to the lowest strictly according to the marks obtained in an exam. These ranged from less than 500 to over 7500. He was thus able to compare the Senior with the Second Wrangler, and he noted the enormous discrepancy often observed, with the Senior frequently achieving double the marks of the Second.

For Galton, this was a demonstration of the enormous range of inherited abilities. His tripartite theory of eminence – capacity, zeal, and power to work hard – is quite general and can be applied to any career. By capacity, he seems to have meant something like what we would now call intelligence or *g*. There was thus no special capacity for mathematics. He seeks to demonstrate this by analyzing the subsequent careers of top Wranglers. He notes that several were also classical scholars, and a few were both Senior Wranglers and the top classical prizemen of the year. Many achieved distinction in areas very different from mathematics, such as law, politics, or becoming headmasters of great schools.

His example of D'Alembert (1717–1783) reinforces this point. Not only was he a mathematician of great distinction, he was even better known as the co-editor with Denis Diderot of the *Encyclopedie*.

Intelligence

Ability in mathematics is widely seen as a marker for intelligence, and disability in mathematics in school is seen as a marker for low intelligence. This is not the place to discuss the general relationship between intelligence and mathematical ability but only to point out that severe disability can co-occur with good to superior IQ scores (Landerl et al., 2004).

In contrast to Galton, Mitchell (1907) noted that "Skill in mental calculation is, owing to the isolation of mental arithmetic already noted, independent of general education: the mathematical prodigy may be illiterate or even densely stupid, or he may be an all-round prodigy and veritable genius" (p. 131). Many expert calculators achieved eminence in a way that suggested exceptional cognitive abilities. These included the mathematicians Euler, Gauss, Aitken, and D'Alembert, and scientists and engineers such as Ampère, Bidder, and Mitchell himself.

Shakuntala Devi (born 1940) is in the Guinness Book of World Records for being able to multiply two 13 digit numbers in 28 seconds. She was tested formally by the psychologist A. R. Jensen, who showed that she was not much better than average on standard IQ tests, and was actually slower than average on some tests of speed of mental processing, which Jensen regards as a reliable measure of intelligence (Jensen, 1990). Dehaene comments that therefore "Devi's calculation were obviously not due to a global speed up of her internal clock: Only her arithmetic processor ran with lightning speed" (Dehaene, 1997).

Moreover, other prodigious calculators seemed to have been men of ordinary or even very low cognitive ability. Dase, who did calculations for Gauss and Schumacher, was "unable to comprehend the first elements of mathematics." One pair of twins with prodigious abilities for calendrical calculation were estimated to have IQs in the 60s and had great difficulties with simple arithmetic (Horwitz, Deming, & Winter, 1969). Mitchell (1907) noted that two prodigious calculators, Fuller (1710?–1790) and Buxton (1702–1772), "were men of such limited intelligence that they could comprehend scarcely anything, either theoretical or practical, more complex than counting" (pp. 98–99).

Hermelin and O'Connor reported a young man who was able to recognize and generate primes of up to five digits four or five times faster than a graduate with a math degree and also factorize these numbers faster and more accurately. What is extraordinary is that the man had a measured IQ of 67 and was unable to speak or understand speech (Hermelin & O'Connor, 1990). See Horn and Masunaga, Chapter 34, for further discussoin of intelligence and expertise.

Memory

"The distinction often made between *memory* and *calculation*, with the implication that the great calculator is simply a little calculator with a big memory, using the same methods as his lesser rivals, is misleading; the process is always (in the "natural calculators") a true calculation, and memory for figures is important only in so far as it stands in the service of calculation" (Mitchell, 1907, p. 132).

Scripture (1891) distinguished "accuracy" and "rapidity" of memory from what he called "association." We would now call the former "working memory" and the latter long-term "semantic memory" (Cappelletti, Kopelman, & Butterworth, 2002). Calculators themselves stressed the importance of both being able to hold many items in mind as they were carrying out calculations, and also knowing many more facts about numbers than the average person.

Working memory

According to Smith (1983), George Parker Bidder (1806–1878), an exceptional calculator and a leading engineer of his time (a collaborator with Robert Stephenson), was the first to explicitly draw attention to working-memory limitations on calculation. Bidder noted that "As compared with the operation on paper, in multiplying 3 figures by 3 figures, you have three lines of 4 figures each, or 12 figures in the process to be added up; in multiplying 6 figures into 6 figures, you have six lines of 7 figures, or 42 figures to be added up." In general, the difficulty in using the mental analogue of the written method increases by something in the order of $n^2 + n$ of the number of digits in an $n \times n$ problem (Smith, 1953, p. 53). For this reason, it very important for the calculator to develop techniques for reducing current load. Given that a three digit number was, for Bidder (and most calculating experts) a single item, a three-digit by three-digit calculation, working from the left (instead of the right, as is normal in the written method), requires no more than five items to be currently maintained. For the problem 358×464, assuming trailing zeros are stored at no cost, Bidder probably worked it out thus:

Although there are many steps, the current load is kept small, and the routine is easy to practice.

Wim Klein (1912–1986), one of the fastest calculators, would write down intermediate results, which, he claimed, speeds up the process, an important element important when there is an audience. Smith (1983) timed him multiplying two five-digit numbers. With writing down, it took 14 seconds, and without, 44 seconds.

Perhaps the most detailed psychological investigation of working memory comes from two studies of German calculating prodigy Rüdiger Gamm. Gamm is able to calculate the 9th powers and the 5th roots with great accuracy, and find the quotient of two primes to 60 decimal places. Even more extraordinary is that he started training for these feats when he was 20 years old. Before then, his mathematical abilities had been unexceptional.

Gamm, again like other experts, is able to solve multi-step problems very quickly and accurately. To solve 68×76 takes

Table 32.1.

Step	*Numbers in memory*	*Number of items to be maintained*	*Calculation*
1	358 464	2	
2	120000	3	400 × 300
3	20000	4	400 × 50
4	140000	3	120000 + 20000
5	3200	5	400 × 8
6	143200	4	140000 + 3200
7	18000	5	60 × 300
8	161200	4	143200 + 18000
9	3000	5	60 × 50
10	164200	4	161200 + 3000
11	480	5	60 × 8
12	164680	4	164200 + 480
13	1200	5	4 × 300
14	165880	4	164680 + 1200
15	200	5	4 × 50
16	166080	4	165880 + 200
17	32	5	4 × 8
18	166112	4	166080 + 32

(adapted from Smith, 1953, p. 54)

seven steps and six intermediate results. After some practice with the task, Gamm was taking around five seconds a problem with a high degree of accuracy. (Two digit squares, by contrast, took him just over a second because they were simply retrieved from memory.) Such a sequence of operations and data handling would put a considerable strain on normal working memory, yet all kinds of expertise show enormous gains in the temporary storage of task-relevant materials: musicians can recall tunes after a single hearing, chess masters can recall positions after a single tachistoscopic presentation as well as the whole game that they have just played, expert waiters can keep in mind the precise orders for up to 20 people without writing them down (at least until the customer has paid). Experts develop a kind of "Long-term Working Memory" (Ericsson & Kintsch, 1995).

"Long-term Working Memory"

As we have seen, one of the barriers to mental calculation is the limited capacity of working memory. Many exceptional calculators use and invent algorithms that minimize the load on working memory. It has also been suggested that one of the consequences of expertise is the ability to exploit the unlimited capacity of long-term memory in the service of the current task (Ericsson & Kintsch, 1995). It is as if experts "develop an ability to use long-term episodic memory to maintain task-relevant materials, rather as computers extend the capacity of RAM by using swap space on the hard drive to create a larger 'virtual memory'" (Butterworth, 2001, p. 12).

Language processing is a more familiar example of prodigious skill after years of daily practice enabling retention of information well beyond the span of short-term working memory. We can effortlessly retain meaningful sentences of 20 words or more, well beyond the span for unrelated words (about six) or words not in our language (about three). Several related accounts of this phenonemon propose cues in working memory for retrieving well-organized domain-specific information in long-term episodic memory (Butterworth, Shallice, & Watson, 1990). Pesenti and colleagues argue that Gamm has learned to use this LTWM facility to maintain task-related mathematical information.

It turned out that computation compared to retrieval of memorized number facts in both Gamm and the controls activated an extensive visual processing system bilaterally. According to the authors, this suggests that "during complex calculation, numbers are held and manipulated onto a visual type of short term representational medium." This contrasts with the more usual claim that "sub-vocal rehearsal is . . . required for mental arithmetic" (Logie, Gilhooly, & Wynn, 1994), but it would explain how it is possible for brain damage to reduce digit span to two and yet allow a patient to reliably add two orally presented three-digit numbers (Butterworth, Cipolotti, & Warrington, 1996). (See this volume for more on LTWM.)

We will see below that Gamm's use of LTWM is supported by analysis of neural acitivity.

Auditory and Visual Working Memory

Mitchell noticed that there seemed to be two types of working memory used in calculation, visual and auditory, depending on how numbers were initially learned. Most children learn about calculating by counting aloud using the names of numbers, names often some years before they understand written numerals (Gelman & Gallistel, 1978). There appear to be three main stages in the development of counting as an addition strategy:

1. *Counting all. For 3 + 5, children will count "one, two, three" and then "one, two, three, four, five" countables to establish the numerosity of the sets to be added, so that two sets will be made visible – for example, three fingers on one hand and five fingers on the other. The child will then count all the objects.*
2. *Counting on from first. Some children come to realise that it is not necessary to count the first addend. They can start with three, and then count on another five to get the solution. Using finger counting, the child will no longer count out the first set, but start with the word "Three," and then use a hand to count on the second addend: "Four, five, six, seven, eight."*
3. *Counting on from larger. It is more efficient, and less prone to error, when the smaller of the two addends is counted. The child now selects the larger number to start with: "Five," and then carries on "Six, seven, eight." (Butterworth, 2005)*

However, many calculators report that their early experiences involved manipulables. Bidder described how he learned multiplication in the following way: "I used to arrange [peas, marbles, or shot] into squares, of 8 on each side, and then on counting them throughout, I found that the whole number amounted to 64" (Quoted by Smith, 1983, p. 212). It is probable that Bidder, like others who had early experience with manipulables, also used a kind of visual coding. Salo Finkelstein (born 1896), who seemed to have had a standard Polish mathematical education, without showing early signs of exceptionality, calculated by visualizing numbers on a freshly washed blackboard. His calculation ability seemed not to have equalled many other prodigies in terms of time or accuracy, but his ability to memorize numbers was. He was able to remember numbers up to about 28 digits following a one-second visual exposure; for 39 digits he needed a four-second exposure. He was adept at repeating in either direction with equal accuracy, which traditionally suggests a visual memory. However, he also used a wide variety of associations for substrings to help him, including numerical facts, such as the fact that 1.41 is the square root of 2, 2,592,000 is the number of seconds in a month, 10,592 is a familiar telephone number, and 2595 is the number of paragraphs of Spinoza's ethics (Smith 1983, Chapter 33).

Dehaene and colleagues (Dehaene & Cohen, 1995) have proposed that multidigit arithmetic of the sort carried out by calculators depends on visualizing the digits on a kind of mental blackboard. There is some evidence that neurological damage can lead to deficits in spatial cognition, which can lead to a kind of spatial "acalculia" where the patient has difficulty in maintaining the digits in columns accurately (Hécaen, Angelergues, & Houillier, 1961).

In the case of Gamm, it was possible to identify the brain areas active during calculation, and hence whether verbal or visual areas were active. It turned out that computation compared to retrieval of memorized number facts in both Gamm and the controls activated an extensive visual processing system bilaterally. According to Pesenti and colleagues, this suggests that "during complex calculation, numbers are held and manipulated onto a visual type of short term representational medium" (Pesenti et al., 2001). This contrasts with the more usual claim that "sub-vocal rehearsal is . . . required for mental arithmetic" (Logie et al., 1994), but it would explain how it is possible for brain damage to reduce digit span to two and yet allow the patient to reliably add two orally presented three-digit numbers (Butterworth et al., 1996).

Domain-specificity in Memory

Gamm had a forward span of 11 digits (controls 7.2, SD = 0.8) and 12 digits backwards (controls 5.8, SD = 0.8), whereas his letter span was in the normal range (Pesenti, Seron, Samson, & Duroux, 1999).

Mondeux (1826–1861), a famous nineteenth-century calculator, was described by a contemporary as never having learned anything besides arithmetic; "Facts, dates, places, pass before his brain as before a mirror without leaving a trace" (quoted by Smith, 1983, p. 294)

Long-Term Working Memory (Ericsson & Kintsch, 1995), deployed by experts, is specific to the domain of expertise; thus, the musician, the chess master, and the waiter will be normal on for example digit span, (Ericsson & Kintsch, 1995). So, as Ericsson and Charness note, "exceptional memory is nearly always restricted to one type of material" (Ericsson & Charness, 1994).

"Management" and "Strengthening" Memories

Solving even a simple arithmetical problem can be broken down into separable components, which will include retrieving arithmetical facts from memory, retrieving procedures for calculating (such as borrowing and carrying), understanding the arithmetical concepts demanded by the problem, and creating a hierarchical set of goals and subgoals appropriate for reaching the solution. Charness and Campbell have shown that, in learning a new algorithm for multiplying double-digit numbers, the memory elements are strengthened by practice, but there is a larger effect from the overall approach to the problem, particularly, from "increased efficiency in managing memory and accessing the next step in the procedure" (Charness & Campbell, 1988)

Convergent evidence for the compositionality of arithmetical task comes from neurological patients, whose arithmetical abilities can be selectively affected in very specific ways: the memories for facts alone can be lost, indeed the memories for facts from each of the four arithmetical operations can be selectively impaired (Cipolotti & van Harskamp, 2001; van Harskamp & Cipolotti, 2001); arithmetical procedures can be lost from memory (Girelli & Delazer, 1996; McCloskey & Caramazza, 1987); and the ability to apply arithmetical principles to problems can be selectively spared or affected (Delazer & Benke, 1997; Hittmair-Delazer, Semenza, & Denes, 1994).

It is, as has been noted above, that mathematical experts and calculating prodigies build up enormous stores of what Scripture called "associations" – numerical facts and procedures.

Perhaps the greatest of recent calculators, Wim Klein acquired the multiplication tables to 100 × 100 "from experience [he] got by factoring." However, he did set out to memorize the table of logarithms up to 150. This training enabled to him to achieve the world record in extracting roots. He could extract the 13th root of a 100-digit number in under two minutes by using a method that requires taking logarithms of the leftmost group of numbers.

Aitken, similarly, had an enormous store of number facts. For him the year 1961 evoked the thoughts 37×53, $44^2 + 5^2$, and $40^2 + 19^2$. He could also recite the first 100 decimal places of π (Hunter, 1962).

Like other calculating prodigies, Gamm taught himself a vast range of number facts. Most of us know our multiplication tables, and perhaps 50 simple additions (Ashcraft, 1995), but Gamm has learned tables of squares, cubes, roots, and so forth. Most of us know a few procedures for working out problems that we cannot retrieve from memory, whereas Gamm has an enormous store of procedures and shortcuts, some of which he has learned from books, others he has worked out for himself. Gamm, again like other experts, is able to solve multi-step problems very quickly and accurately. To solve 68×76 takes seven steps and six intermediate results. After some practice with the task, Gamm was taking around five seconds per problem with a high degree of accuracy. (Two digit squares, by contrast, took him just over a second because they were simply retrieved from memory.)

Motivation and Instruction

"Zeal" and "Inclination"

Most exceptional calculators seem to have been obsessed with numbers from the time they began to count. Jedediah Buxton kept a record of all the free drinks he received from demonstrating his calculating prowess, Thomas Fuller counted the hairs in a cow's tail, and Arthur Griffiths (1880–1911) kept track of the grains of corn he fed to the chickens: 42,173 over three years (Smith, 1953, p. 277). Srinivasa Ramanujan (1887–1920), a prodigious calculator and, according to G. H. Hardy (Hardy, 1969), a natural mathematical genius in the class of Euler or Gauss, would work at mathematics in the mornings before work, often having stayed up all night working on problems.

Calculators from an early age develop a kind of intimacy with numbers. When Bidder was learning to count to 100, the numbers became "as it were, my friends, and I knew all their friends and acquaintances" (Smith, 1983, p. 5). Klein told Smith that "Numbers are friends for me, more or less. It doesn't mean the same for you, does it, 3,844? For you it's just a three and an eight and a four and a four. But I say, 'Hi, 62 squared.'" In a famous story, Hardy visited Ramanujan in hospital and mentioned that the taxi in which he had come was number 1729, "A rather dull number." "No, Hardy! It is a very interesting number. It is the smallest number expressible as the sum of two cubes in two different ways" (C. P. Snow in his introduction to [Hardy, 1969]).

In some cases, there is an incident that awakens the interest. For Aitken, a teacher "chanced to say that you can use the factorization to square a number: $a^2 + b^2 = (a + b)(a - b) + b^2$. Suppose you had 47 – that was his example – he said you could take b as 3. So $(a + b)$ is 50 and $(a - b)$ is 44, which you can multiply together to give 2200. Then the square of b is 9, and so, boys, he said, 47 squared is 2209. Well, from that moment, that was the light, and I never went back" (Hunter, 1962).

In the case of Gamm, he said that at school he was "very bad at arithmetic" because the teachers never explained the concepts in ways he could understand. As a result he lost interest in mathematics until about the age of twenty, when he came across an algorithm for calendrical calculation. He practiced it for fun, and then entered for a TV competition where he could win bets by solving various calculations. See Zimmerman, Chapter 39, for more on motivational factors in the development of expertise.

The Role of Practice – 10,000 Hours

The highest level of expertise in violinists studied by Ericsson et al. (1993) requires 10,000 hours of practice (by the age of 20 years). In general, the level of expert performance was related directly (monotonically) to the amount of practice. Similarly, expert calculators spend a great deal of time learning numerical facts and procedures, though the exact amount has never been properly quantified. In preparation for the T.V. program, Gamm started to train

Table 32.2.

	638 × 823	7286 × 5397
Inaudi	6.4 sec	21 sec
Diamandi	56 sec	2 min 7 sec
Best cashier	4 sec	13 sec

up to four hours a day, learning number facts and calculation procedures. He now performs professionally. His expertise is rare enough to be a cause of wonder (the usual definition of prodigy).

Some of the best evidence for the pure effects of practice comes from an experiment carried out by Binet in which he compared the performance of two professional calculators, Inaudi and Diamandi, with cashiers from the Bon Marché department store in Paris, who had had 14-years experience of calculating (there were no mechanical calculators available in the 1890s), but who, presumably, showed no special early gift for mathematics. He compared how long it took them to carry out multidigit multiplications. Although the timing was about as accurate as it could have been without voice-activated relays, it is far from clear that the conditions were the same for each subject, and the different subjects were mostly given different problems to solve. However, they were given one identical 3-digit × 3-digit and one 4-digit × 4-digit problem. For these stimuli, the best cashier was better than either calculator: As can be seen at least one cashier was better than the professionals, but all were better than Binet's students. See Ericsson, Chapter 38, on the roles of experience and deliberate practice.

Education

Ericsson and colleagues have stressed the importance in reaching high levels of expertise of "optimal environments for . . . children" and cite examples of parents who have designed such environments irrespective of objective evidence for innate talent in the children (Ericsson & Charness, 1994). One can think of the Polgar sisters in chess, the Williams sisters in tennis, Tiger Woods in golf, and so on; Mozart grew up in a musical household, and Picasso's father was himself a painter.

This optimal environment encourages "Deliberate practice" with its "individualized training on tasks selected by a qualified teacher" and its careful monitoring and feedback (Ericsson, Krampe, & Tesch-Römer, 1993).

However, there are numerous reports of calculating experts who had little education and were entirely, or almost entirely, self-taught. Zerah Colburn (1804–1840) was able at the age of six to calculate the number of seconds in 2,000 years (9,139,200) but "unable to read and ignorant of the name or properties of one figure traced on paper" (Scripture, 1891, p. 13). Even as an adult, "he was unable to learn much of anything, and incapable of the exercise of even ordinary intelligence or of any practical application" p. 16). Scripture inferred that "calculating powers . . . seemed to have absorbed all his mental energy."

Vito Mangiamele (born 1827) was the son of a shepherd who was unable to give the boy any instruction. According to Scripture, "By chance it was discovered that by methods peculiar to himself, he resolved problems that seemed at the first view to require extended mathematical knowledge" (p. 17), for example, "What satisfies the condition that its cube plus five times its square is equal to 42 times itself increased by 40?" ($x^3 + 5x^2 - 42x - 40 = 0$). He found the answer to this (5) in less than a minute when he was ten-years old.

Zacharias Dase (1824–1861) was an extraordinary calculator who, for a time, assisted Gauss in calculating tables. One distinguished mathematician credited him with "extreme stupidity." a view that seemed to be held also by his mathematician collaborators. He knew no geometry and never mastered a word of another language. "He had one ability not present to such a great degree in other ready reckoners. He could distinguish some thirty objects of a similar nature in a single moment as easily as other people can recognise three or four. The rapidity with which he would name the number of sheep in a herd, or books in book-case, or window-panes in a large house, was even more remarkable than the accuracy of his mental calculations" (Scripture, 1891, p. 20). According to Mitchell, he "could count some thirty objects at a glance" (p. 142), though it is not clear what this had to do with his calculating prowess.

Genetics

Galton's account of the parents and offspring of men of eminence did not examine the potential social and educational effects of growing up in a talented and well-connected family. Of course, there will be cases like D'Alembert, and those above, who have achieved eminence despite an apparently unhelpful upbringing.

Genetic studies support the idea of an innate domain-specific system for at least simple mathematics. A recent twin study of mathematical abilities showed that the concordance rates were 0.73 for monozygotic and 0.56 for dizygotic pairs (Alarcon, Defries, Gillis Light, & Pennington, 1997). Looking at the selective deficit of mathematical ability, dyscalculia, of the dyscalculic probands, 58% of monozygotic co-twins and 39% of dizygotic co-twins were also dyscalculic. In a family study, it was found that approximately half of all siblings of children with dyscalculia are also dyscalculic, with a risk five to ten times greater than for the general population (Shalev & Gross-Tsur, 2001).

Another line of research has attempted to assess whether sex-linked characteristics contribute to mathematical expertise. Benbow and colleagues have found in a host of studies a significant advantage for talented 12- to 13-year-old boys over girls at the upper end of the ability range, as measured by SAT-M (Scholastic Aptitude Test – Mathematics), whereas SAT-V (Verbal) showed no comparable difference (see Benbow, 1988, for a review). Benbow argues that the sex difference cannot be explained in terms of "environmental" hypotheses to do with attitudes, confidence, or teaching. She argues rather that a combination of biological differences between the sexes is the cause, in particular a more bilateral neural representation of cognitive functions in the female brain (see next section).

The differences between boys and girls in SAT-M performance appears to follow from the much larger variance in boys' scores, which would allow reliable differences at the top end of the range even if the mean score for girls were higher than for boys (Becker & Hedges, 1988). When one looks at the means, girls in England easily outperform boys in all subjects at all ages. There is one exception to this general rule: mathematics. Girls are only just outperforming boys (DfES, 2002).

On the other hand, Geary (Geary, 1996) reviewed a wide range of industrialized countries and showed that boys, on average, still outperform girls in mathematical problem solving. However, even in the USA at 17 years the *average* difference between boys and girls is still only 1%. The most recent cross-national comparisons using the same tests in all countries, the Third International Maths and Science Survey (TIMSS) (Keys, Harris, & Fernandes, 1996), reinforces the overall picture that in most countries, including the USA, there is no statistical difference in the means, though there are enormous differences among countries, suggesting that educational and cultural factors are vastly more important than gender in the acquisition of mathematical skills.

Brain Systems for Mathematical Expertise

There is now extensive evidence that routine numerical tasks involve a fronto-parietal network (Pesenti et al., 2000), where the parietal components, perhaps especially the left intraparietal sulcus, are relatively specialized for numbers (Dehaene, Piazza, Pinel, & Cohen, 2003). It is certainly the case that damage to the left parietal lobe can severely affect calculation (Cipolotti & van Harskamp, 2001), though almost nothing is known about its effect on other mathematical domains.

More complex calculation in relatively non-expert subjects established that the neural basis of simple retrieval (e.g., $3 \times 4 = ?$), relative to a reading control, "engaged a left parieto-precentral circuit representing a developmental trace of a finger-counting representation that mediates, by extension, the numerical knowledge in adult," plus a naming network including the left anterior insula and the right cerebellar cortex (Zago et al., 2001). On the other hand, complex computation (e.g., 32×24) engaged, additionally, a left parieto-superior frontal network for holding multi-digit numbers in visuospatial working memory along with bilateral inferior temporal gyri, which is implicated visual mental imagery. Correlated activity in the left intraparietal sulcus and the precentral gyrus "may reflect the involvement of a finger movement representation network" in the calculation process. This is not to say that these skilled adults are counting on their fingers, but it may be that the childhood use of fingers in learning to calculate somehow creates the neural substrate for later acquisition of numerical knowledge (Butterworth, 1999).

There have been very few studies of the brain systems of expert calculators. Benbow, O'Boyle, and colleagues (e.g., Alexander, O'Boyle, & Benbow, 1996; O'Boyle, Benbow, & Alexander, 1995; O'Boyle, Gill, Benbow, & Alexander, 1994; Singh & O'Boyle, 2004) have investigated mathematically gifted children and adolescents, with special reference to gender and brain organization. In general, they have found more right-hemisphere involvement in a range of tasks, though, curiously, mathematical tasks themselves have not been studied. Pesenti and colleagues have published data on the brain of an expert calculator carrying out mathematical tasks (Pesenti et al., 2001).

In a functional neuroimaging study, Pesenti and colleagues found that the prodigy Gamm's calculation processes recruited the same neural network as previously observed for both simple and complex calculation (Zago et al., 2001), *plus* a system of brain areas implicated in episodic memory, including right medial frontal and parahippocampal gyri, whereas those of control subjects did not (Butterworth, 2001; Pesenti et al., 2001). Functional brain imaging has established that speech-based working-memory storage, of the kind that supports standard digit-span tasks, involves the perisylvian language areas (Paulesu, Frith, & Frackowiak, 1993). So Gamm's activations here are quite different. As noted above, it has been suggested that experts develop a way of exploiting the unlimited storage capacity of long-term memory to maintain task-relevant information, such as the sequence of steps and intermediate results needed for complex calculation, whereas the rest of us still rely on the very limited span of working memory (Ericsson & Kintsch, 1995). Gamm's activations are consistent with his having developed LTWM for arithmetical calculations (Butterworth, 2001). See also Hill and Schneider, Chapter 37, concerning brain changes with expertise development.

Conclusions

Our starting point was Galton's tripartite theory of eminence: capacity, zeal, and the ability to do a very great deal of hard work.

Starting with capacity, it is clear that cases of individuals with exceptional mathematical, and especially calculating, ability show enormous variety of cognitive abilities. Some are highly intelligent, others averagely intelligent, yet others are classed by their peers (before standardized IQ testing) as stupid. So the kind of general intellectual capacity supposed by Galton does not seem to apply here. Nor does our survey support Gardner's (1983) idea of a distinct "logical-mathematical" intelligence, since many prodigies seem no better than average, and indeed many are much worse than average, in reasoning.

Zeal seems to be a characteristic common to all the prodigies described here. They are obsessed with numbers, treat them as familiar friends, and actively seek closer acquaintance with them.

They also seem to spend a great deal of time thinking and learning about numbers, presumably for many hours a day: all seem to have the capacity for very hard work. Extensive practice has an effect on memory, as would be expected, and it is quite specific. Exceptional calculators have acquired enormous repertories of arithmetical facts and procedures, sometimes deliberately and sometimes by virtue of working with numbers so much. In some cases, excellent arithmetical memory goes hand in hand with very poor memory for other materials. Working-memory is frequently cited as a serious limitation on complex mental calculation, and eminent calculators learn or devise tricks to reduce working-memory load.

Is their exceptional ability confined to mathematics? Whereas some seem to excel only in calculation, others have shown eminence in fields other than mathematics. Although there appears to be specialized brain systems for numerical processing in the parietal lobes, which have an innate basis, this may have little or nothing to do with exceptional ability. This is confirmed by neuroimaging studies: exceptional calculators such as Gamm seem not to be activating the usual brain regions differently, but rather recruiting new regions outside the parietal lobes to support the current task. There is now ample evidence for activity-dependent plasticity: that is, that the functioning, and even the structure, of brain systems is shaped by practice and experience (Amunts et al., 1997; Pascual-Leone & Torres, 1993; Schlaug, Jancke, Huang, Staiger, & Steinmetz, 1995; Schlaug, Jancke, Huang, & Steinmetz, 1995).

Ericsson and Charness (1994) have stressed the role of systematic teaching for promoting the deliberate practice needed for the highest levels of expertise. This, at least in part, is because deliberate practice is not in itself rewarding. There are, in the biographies of mathematical prodigies, many counterexamples to this claim, where precocity in mathematics could be nurtured in a systematic way, whereas others appear to have acquired exceptional mathematical skills despite very unhelpful early conditions.

It may be that finding solutions to mathematical problems is, for the zealous, intrinsically rewarding. It may also be that the domain of mathematics is so ordered that it is propitious for unsupervised learning since it is easy to check an answer by using a different method. Many prodigies report external rewards also – amazing their friends and family. This may be especially relevant in the savant, or near-savant cases, where there may be few ways to gain the admiration of other people. Perhaps this is why parallels between music and mathematics are noticed. Both have intrinsic rewards that are propitious for unsupervised learning. In music, one can hear whether something sounds right or not – there is harmony or there is discord. And there are external rewards that do not require a teacher, namely, that other people readily appreciate good playing or singing.

Finally, are exceptional calculators born or made? There is ample evidence for zeal and hard work, and it may be that we are born with dispositions toward them. Charles Darwin, in a letter to Galton, wrote “I have always maintained that excepting fools, men did not differ much in intellect, only in zeal and hard work; I still think this an *eminently* important difference” (quoted by Ericsson & Charness, 1994). It may also be the case that some of us are born with a disposition to enjoy or even be obsessed with an orderly domain like mathematics. However, there is no evidence at the moment for differences in innate specific capacities for mathematics.

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