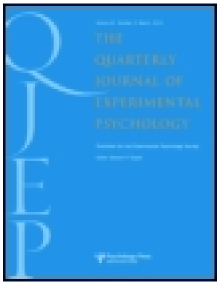

The Quarterly Journal of Experimental Psychology
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/pqje20

Understanding the real value of fractions and decimals

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Available online: 18 Jul 2011

To cite this article: Teresa Iuculano & Brian Butterworth (2011): Understanding the real value of fractions and decimals, The Quarterly Journal of Experimental Psychology, DOI:10.1080/17470218.2011.604785

To link to this article: http://dx.doi.org/10.1080/17470218.2011.604785

THE QUARTERLY JOURNAL OF EXPERIMENTAL PSYCHOLOGY
i*First*, 1–11

Psychology Press
Taylor & Francis Group

Rapid communication

Understanding the real value of fractions and decimals

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Understanding fractions and decimals is difficult because whole numbers are the most frequently and earliest experienced type of number, and learners must avoid conceptualizing fractions and decimals in terms of their whole-number components (the "whole-number bias"). We explored the understanding of fractions, decimals, two-digit integers, and money in adults and 10-year-olds using two number line tasks: marking the line to indicate the target number, and estimating the numerical value of a mark on the line. Results were very similar for decimals, integers, and money in both tasks for both groups, demonstrating that the linear representation previously shown for integers is also evident for decimals already by the age of 10. Fractions seem to be "task dependent" so that when asked to place a fractional value on a line, both adults and children displayed a linear representation, while this pattern did not occur in the reverse task.

Keywords: Fractions; Decimals; Mathematical cognition; Number line; Mathematical education.

A variety of studies have found that children have difficulty understanding fractions and decimals (Bright, Behr, Post, & Wachsmuth, 1988; Ni & Zhou, 2005). In these tasks, children treat fractions and decimals in terms of their whole-number components, sometimes termed the "whole-number bias" (Ni & Zhou, 2005). This is not surprising since learners have to make a large conceptual leap from thinking of numbers as integers (Smith, Solomon, & Carey, 2005). The transition from a system where numbers are used for counting to one that uses proportions can be problematic in late adolescence (Hoyles, Noss, & Pozzi, 2001) and even for educated adults (Bonato, Fabbri, Umiltà, & Zorzi, 2007; but see Schneider & Siegler, 2010). In the study by Bonato et al. (2007), when participants were asked to compare the real values of two fractions—for example, 1/8 versus 1/7—they typically used a strategy that relied on comparing the integer values of the numerator or the denominator rather than computing the real value of the fraction. A similar whole-number bias is evident in children aged 8–11 years when asked to select the larger of two decimal number pairs: .65 versus .8 (Rittle-Johnson, Siegler, & Alibali, 2001; Smith et al., 2005).

This is analogous to the comparison of double-digit numbers, which are also processed in a componential manner—that is, responses are quicker and more accurate if both the unit and the decade

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http://www.psypress.com/qjep
DOI:10.1080/17470218.2011.604785

of one number are larger—for example, determining whether 57 is larger than 42 is easier than 62 versus 47 (Nuerk, Weger, & Willmes, 2001).

Although adults prefer to rely on the comparison of the single components, the real value of fractions can be accessed (Meert, Grégoire, & Noël, 2009), and neuroimaging studies have shown that specific populations of neurons in the parietal and frontal cortices are tuned to the real value of fractional stimuli (Ischebeck, Schocke, & Delazer, 2007; Jacob & Nieder, 2009).

So far, the experimental studies investigating the representation and processing of fractions have used number comparison paradigms where a componential strategy is effective and is presumably easier than calculating the real value of two fractions, storing the value in working memory, and then comparing the two results. Thus performance may not directly reflect the underlying representations.

In the present study, we aim to control for these potential confounds by implementing a number line task in which participants have to place a mark on a physical line to denote the magnitude of the number presented. This is a variant of the classic "thermometer task" used for the quick clinical evaluation of numerical competence in neuropsychological patients (van Harskamp, Rudge, & Cipolotti, 2002). Also, it is commonly used in the educational and developmental literature, usually with subdivisions marked and often as a forced-choice task (Bright et al., 1988; Rittle-Johnson et al., 2001), and most recently to investigate the representational change of magnitudes given in proportional notations (Opfer & DeVries, 2008). Additionally, we also use the opposite manipulation where participants see a hatch mark on a line and are asked to report the numerical value (Siegler & Opfer, 2003). Our procedure differs from these previous studies that exploit number line tasks in the fact that we use a computerized version enabling us to record both accuracy and reaction times.

Moreover, the novelty of this study lies in using a paradigm that consists of four different types of numerical representation, not only fractions (e.g., 1/3, Bright et al., 1988) or decimals (e.g., 0.33, Rittle-Johnson et al., 2001), but also proportional equivalents as two-digit whole numbers (e.g., 33) or as money (33p). Finally, to reduce an advantage for a componential strategy, we used fractions without common components, and we define the end points of the number line by 0 and 1 for fractions and decimals, rather than by fractional stimuli ("0/3" and "3/3"; Opfer & DeVries, 2008).

Moreover, we test adults and children, enabling us to plot developmental trends of the proportional numerical reasoning abilities for these types of numbers.

Method

Participants

Adults. A total of 18 university students from a variety of academic backgrounds[1] (7 males, mean age = 24.3 years; *SD* = 6.74) took part. All except one were right-handed, and all reported normal or corrected-to-normal vision.

Children. A total of 19 normal-achieving children in their sixth year of schooling recruited from two different middle schools in London (7 males, mean age = 10.83 years; *SD* = 0.23) took part. All children were screened for learning disabilities on a range of numerical and cognitive tasks.

[1] The level of mathematical education was taken into account. Participants were divided into three subgroups: no maths A-level, maths A-level, and maths at university level. The slope (β_1 value) of the regression equation $Y = \beta_0 + \beta_1 x$ of the linear model was taken as an index of representational acuity. The more the β_1 value deviates from 1, the less accurate the estimate. A 4 (notation condition) × 3 (education groups) mixed model analysis of variance was performed. In the number to position (NP) task, there was no main effect of notation condition, $F(3, 45; \varepsilon = .672) = 0.796$, $p = .461$, $\eta^2 = .05$, no main effect of group, $F(2, 15) = 2.553$; $p = .111$, $\eta^2 = .25$, and no significant interaction, $F(3, 45; \varepsilon = .672) = 0.340$, $p = .851$, $\eta^2 = .043$. Similarly, in the position to number (PN) task, none of the effects was significant: notation condition, $F(3, 42; \varepsilon = .339) = 0.225$, $p = .647$, $\eta^2 = .016$, group, $F(2, 14) = 0.893$, $p = .432$, $\eta^2 = .113$, and interaction, $F(3, 42; \varepsilon = .339) = 1.22$, $p = .325$, $\eta^2 = .148$. The three subgroups were therefore merged into one group for subsequent analyses.

Apparatus

Two complementary estimation tasks were used, number to position (NP) and position to number (PN; see Siegler & Opfer, 2003).

In the NP task, participants were instructed to move the cursor to the desired position using a mouse on a fixed-length line presented on the computer screen. The number-line coordinates for participant responses were recorded in terms of pixel count along the length of the line. In the PN task, participants were required to type numbers into onscreen boxes. All participants were tested using the same computer (model: Asus m68 00n wide-screen), which has a dot pitch of 0.282 mm.

Each problem involved a line with the left end labelled "0" and the right end labelled "1" or "100" or "£1" depending on the notation condition. The right label changed to conceptually match the notation session, whilst preserving the scaling. The left to right orientation was similar to that in previous studies (Siegler & Opfer, 2003; van Harskamp et al., 2002) and is thought to reflect the orientation of the mental number line in Western subjects (Dehaene, Bossini, & Giraux, 1993).

Four stimulus notation conditions were used in the NP task: *fractions* (e.g., 1/4), *integers* (e.g., 25),[2] *decimals* (e.g., 0.25), and *money* (e.g., 25p). The numerical values of the fractions were all <1 and were evenly distributed on the number line: 1/20, 1/9, 1/6, 1/5, 2/9, 1/4, 2/7, 1/3, 2/5, 4/9, 1/2, 4/7, 3/5, 13/20, 5/7, 3/4, 7/9, 5/6, 6/7, 19/20. The stimuli for the decimals and money notation conditions were selected so that they matched an approximation of the numerical values of the fraction stimuli (e.g., 0.1 for 1/9). Stimuli for the integers condition matched in scale the numerical values of the fraction stimuli (e.g., 25; Figure 1A).

In the PN task, the same four notation conditions were used with marks on the line corresponding to the numerical values in the NP task. For instance, in the fractions condition, participants would have to assess the fractional value of a mark corresponding to 1/4, by freely choosing their estimate from the infinite spectrum of possible fractions (Figure 1B).

Procedure

In the NP task, a line marked with the minimum and maximum values at either end was presented in the centre of the screen simultaneously with a target, centrally positioned on the top of the line. Participants were required to indicate the corresponding position of the target on the line. Participants responded by selecting the appropriate position using a mouse whose cursor starting point was not fixed (Figure 1A).

In the PN task, participants were presented with the same number-line as that in the NP task, with a hatch mark indicator bisecting the line at locations that corresponded to the numerical values used in the NP task. Participants were asked to estimate the value corresponding to the marked position on the line expressed as a fraction, integer, decimal, or money amount (Figure 1B). Participants responded by typing their answer into a box on the screen.

Participants responded at their own pace and were allowed to correct their responses. This study was composed of a single block for each of the four notation conditions (20 stimuli each). Stimulus presentation was randomized within each set; order of tasks and notation conditions were counterbalanced across participants.

Results

Regression analyses

Regression analyses were performed on participants' mean estimates[3] plotted against the actual values of the target stimuli separately for each task.

To identify the best fitting model, paired-sample t tests were applied on the residuals of the regression models of interest: linear and logarithmic for the NP task; linear and exponential for the PN task (Siegler & Opfer, 2003).

[2] By "integer" we mean here non-negative whole numbers.

[3] Individual estimates displayed identical results to the group mean (e.g., Moeller, Pixner, Kaufmann, & Nuerk, 2009).

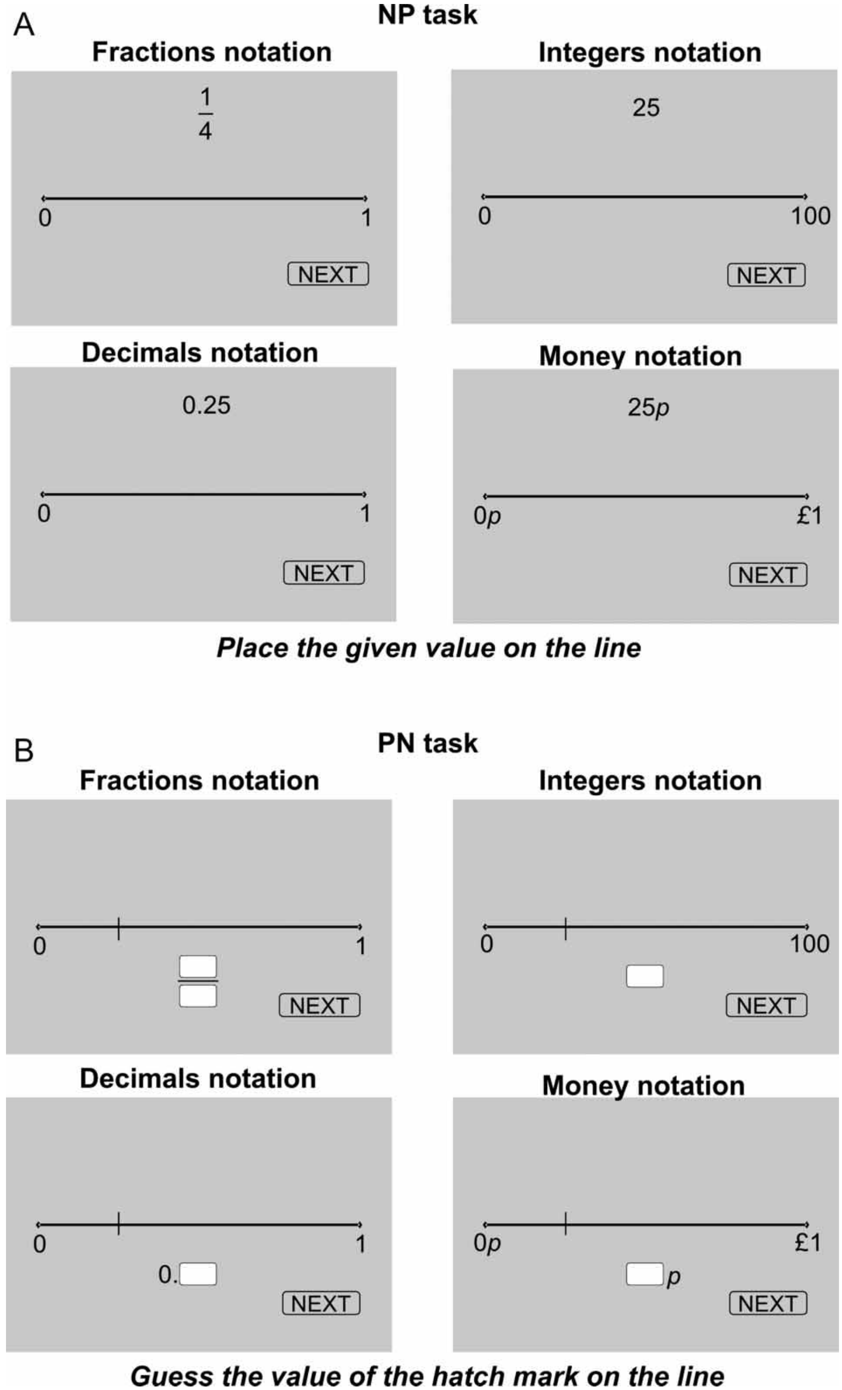

Figure 1. *Example stimuli for both tasks: (A) number to position (NP), and (B) position to number (PN).*

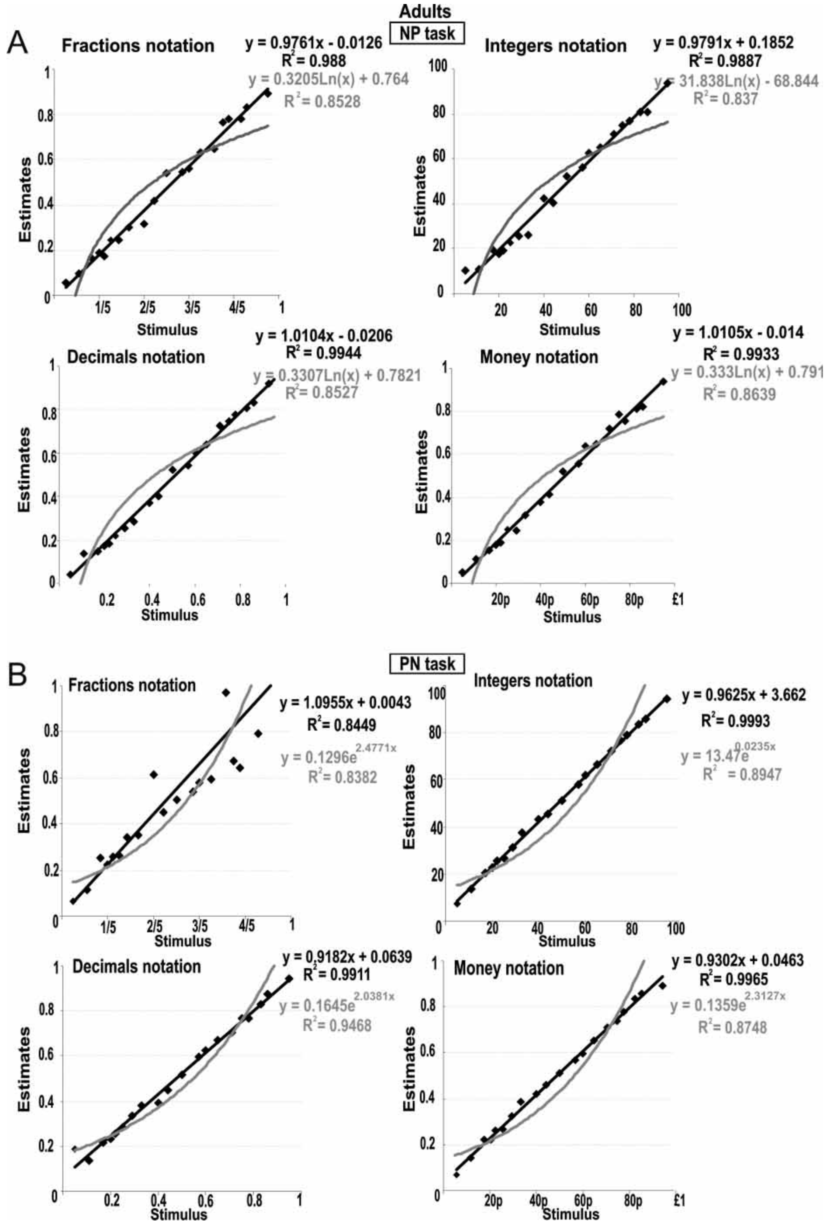

Figure 2. Adults. *Average location of estimates regressed against value of the stimulus for each of the four notation conditions for both tasks: (A) number to position (NP); (B) position to number (PN). In black: equation and best fitting line of the linear model. In grey: (A) equation and best fitting line of the logarithmic model; (B) equation and best fitting line of the exponential model. (A) Top left: in the fractions notation condition, all stimulus estimates fit well on the linear function (*y = x*) with no distinction between familiar and unfamiliar fractions (e.g., 1/2 vs. 3/5).*

Adults. In the NP task, the linear model was significantly the best model for all notation conditions: fractions, $t(19) = -2.74$, $p < .05$; integers, $t(19) = -4.6$, $p < .001$; decimals, $t(19) = -5.55$, $p < .001$; and money, $t(19) = -5$, $p < .001$ (Figure 2A).

In the PN task, the linear model was the best fitting model for integers, $t(19) = -4.24$, $p < .001$; decimals, $t(19) = -2.61$, $p < .05$; and money, $t(19) = -4.05$, $p < .01$, but not for fractions, $t(19) = -0.62$, $p = .54$; Figure 2B).

Children. In the NP task, the linear model was significantly better than the logarithmic model for all notation conditions: fractions, $t(19) = -3.6$, $p < .01$; integers, $t(19) = -4.9$, $p < .001$; decimals, $t(19) = -4.4$, $p < .001$; and money, $t(19) = -4.67$, $p < .001$ (Figure 3A).

In the PN task, the linear model was the best fitting model for integers, $t(19) = -4.24$, $p < .001$; decimals, $t(19) = -2.65$, $p < .05$; and money, $t(19) = -4.52$, $p < .001$; but not for fractions, $t(19) = -1.24$, $p = .23$ (Figure 3B).

Accuracy and reaction time analyses

A 4 (notation condition) × 2 (age group) mixed model analysis of variance (ANOVA) was performed. Pairwise comparison analyses were adjusted for multiple comparisons (Bonferroni corrected). Due to violations of sphericity, Greenhouse–Geisser corrections were applied to all factors with more than two levels (Keselman & Rogan, 1980).

Accuracy. Accuracy analyses were performed on deviants. A deviant was defined as the absolute value of the subtraction between the participants' response and the target value (in pixels on the line).

In the NP task, the analyses revealed a main effect of notation condition, $F(3, 114; \varepsilon = .543) = 89.113$, $p < .001$, $\eta^2 = .701$: The fractions notation was significantly less accurate than the other notations ($p < .001$). There was also a main effect of group, $F(1, 38) = 57.194$; $p < .001$, $\eta^2 = .601$, and a significant interaction, $F(3, 114; \varepsilon = .543) = 30.81$; $p < .001$, $\eta^2 = .448$: The children group displayed lower accuracy than the adults group in the fractions and decimals notations ($p < .005$; Figure 4A).

In the PN task, both main effects were significant in the same direction as that for the NP task: notation condition, $F(3, 114; \varepsilon = .355) = 119.57$, $p < .001$, $\eta^2 = .759$, and group, $F(1, 38) = 89.682$, $p < .001$, $\eta^2 = .702$ (Figure 4B). The interaction was also significant, $F(3, 114; \varepsilon = .355) = 56.734$; $p < .001$, $\eta^2 = .599$: The children group was significantly less accurate than the adults group in all notation conditions, except decimals ($p = .117$; Figure 4B).

Reaction times (RTs). These analyses were performed on median RTs.. In the NP task, both main effects were significant: notation condition, $F(3, 114; \varepsilon = .679) = 60.813$, $p < .001$, $\eta^2 = .615$—the fractions notation was significantly slower than the other notations ($p < .001$)—and group, $F(1, 38) = 8.011$, $p < .01$, $\eta^2 = .174$. The interaction was also significant, $F(3, 114; \varepsilon = .679) = 10.291$, $p < .001$, $\eta^2 = .213$: The two groups differed in their speed of responses in all notation conditions, except fractions ($p = .146$; Figure 5A).

In the PN task, both main effects were also significant: notation condition, $F(3, 114; \varepsilon = .597) = 145.35$, $p < .001$, $\eta^2 = .793$—the fractions notation was performed significantly slower than the other notation conditions ($p < .001$)—and group, $F(1, 38) = 45.297$, $p < .001$, $\eta^2 = .544$, while the interaction was not significant, $F(3, 114; \varepsilon = .597) = 2.881$, $p = .069$, $\eta^2 = .07$ (Figure 5B).

Discussion

In this study, we investigated how children and adults understand and represent the real value of four different types of multidigit number notations: fractions, integers, decimals, and money. It has been proposed that children are characterized by several mental representations of numbers (Berteletti, Lucangeli, Piazza, Dehaene, & Zorzi, 2010), and that the "representational starting kit" is characterized by a logarithmic mental representation of numbers that progresses to a linear representation over the course of development (Siegler & Opfer, 2003). Since previous investigations have only used natural numbers from 0 or 1 upwards, we asked whether this transition applies to alternative ranges and types of numbers.

Previous research has found that fractions and decimals are difficult for children and even well-educated adults (Bonato et al., 2007), so it is

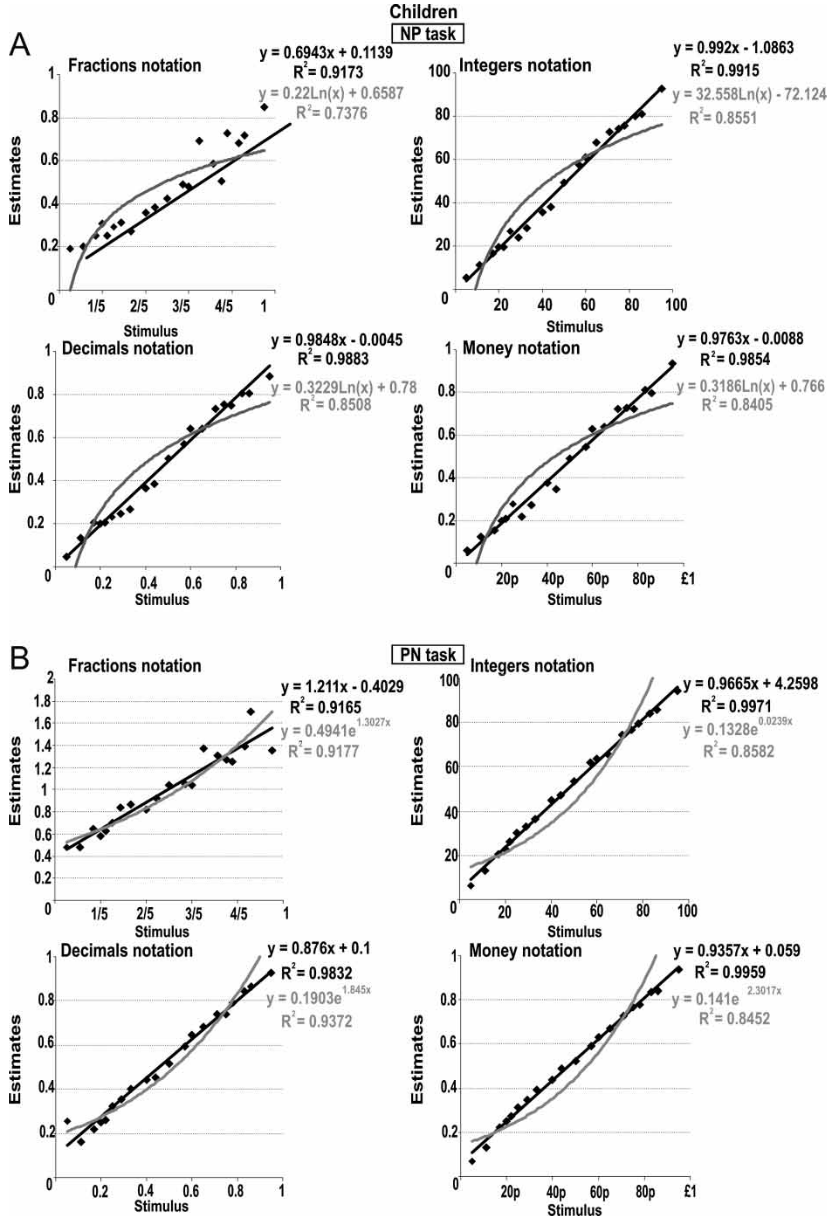

Figure 3. Children. *Average location of estimates regressed against value of the stimulus for each of the four notation conditions for both tasks: (A) number to position (NP); (B) position to number (PN). In black: equation and best fitting line of the linear model. In grey: (A) equation and best fitting line of the logarithmic model; (B) equation and best fitting line of the exponential model. (A) Top left: in the fractions notation condition, all stimulus estimates fit well on the linear function* (y = x) *with no distinction between familiar and unfamiliar fractions (e.g., 1/2 vs. 3/5).*

possible that although 10- to 11-year-olds have linear representations for integers (Siegler & Opfer, 2003), they may still represent the value of fractions and decimals in a nonlinear manner. Indeed, it has been shown that 8-year-olds displaying a logarithmic representation for integers are

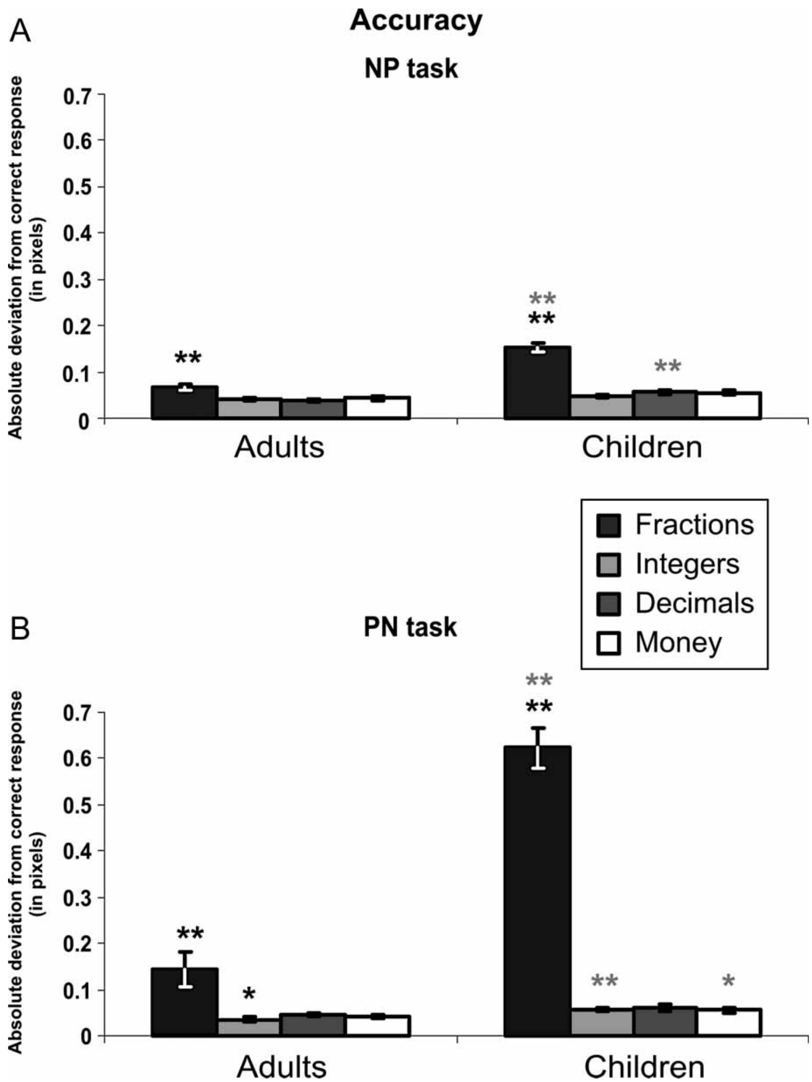

Figure 4. *Accuracy in terms of the absolute deviation from correct responses (in pixels) plotted against age group for both tasks: (A) number to position (NP); (B) position to number (PN). Black asterisks represent significant differences between notation conditions: *p < .05, **p < .005. Grey asterisks represent significant differences between groups: *p < .05, **p < .005. Error bars indicate 1 standard error of the mean (*SEM*).*

facilitated in estimation tasks using fractions with a fixed numerator or denominator (Opfer & DeVries, 2008). No studies thus far have investigated the development of fraction and decimal representations in children who display a linear representation for integers and were formally introduced to the concept of fractions in school.

Our study tested the possible models of number line representations—linear, logarithmic, and exponential—in two number line estimation tasks, which used a variety of numerators and denominators. For the NP task, the best fitting model for both groups was linear, and never logarithmic for any notation including fractions. Thus, even if a developmental transition exists from a compressed (i.e., logarithmic) to a linear representation of numerical values, it is already implemented by the age of 10 for all number notations.

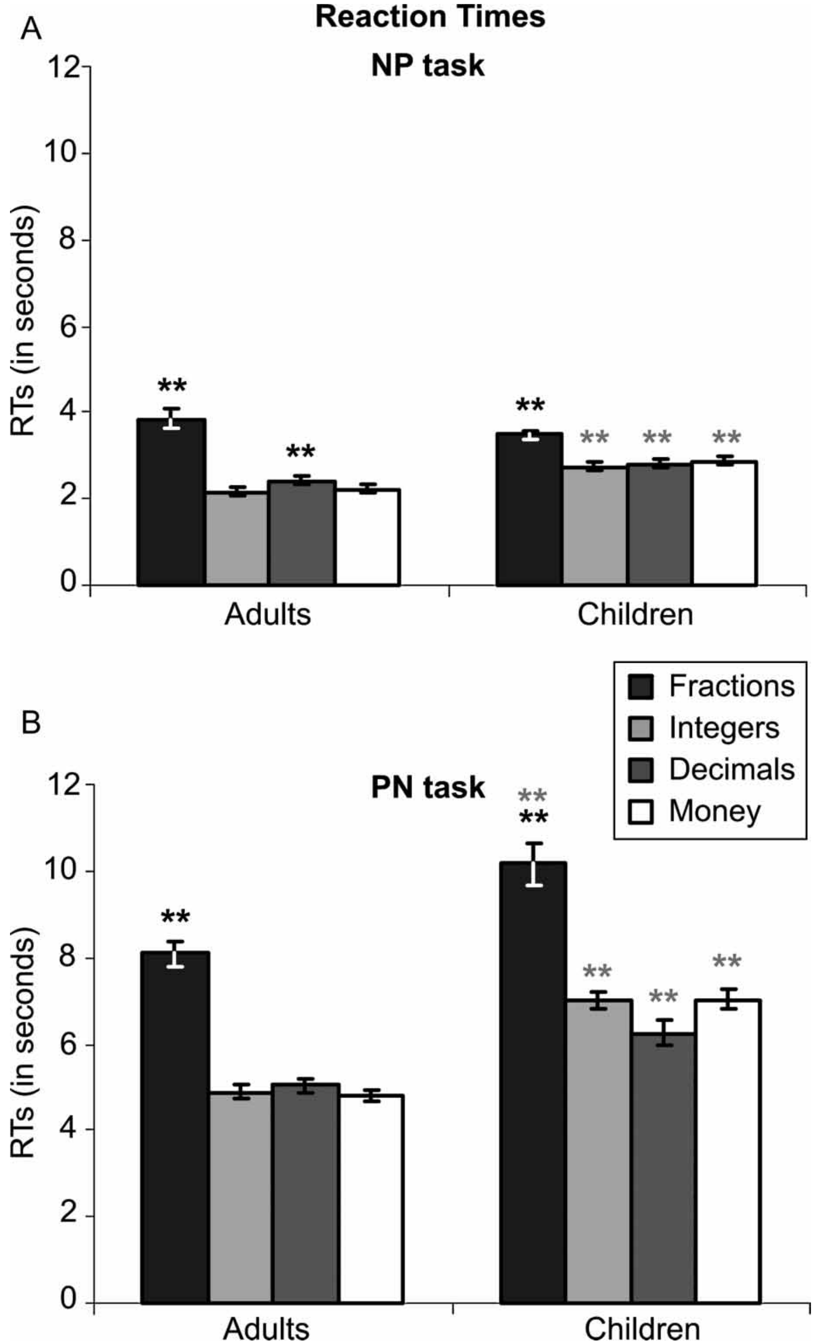

Figure 5. *Reaction times (seconds) plotted against age group for both tasks: (A) number to position (NP); and (B) position to number (PN). Black asterisks represent significant differences between notation conditions: *p < .05, **p < .005. Grey asterisks represent significant differences between groups: *p < .05, **p < .005. Error bars indicate 1 standard error of the mean (*SEM*).*

Additionally, we investigated the accuracy of numerical estimation by measuring actual deviations from correct responses. In the NP task, we showed high accuracy in all notations (deviation from correct responses range between 0.039 and 0.15 pixel counts). However, the mapping of fractions led to significantly more errors than the other notations, even for adults, whilst accuracy for decimals was nearly the same as that for the integers and money notations (Figure 4A). Furthermore, the processing of fractional stimuli took twice as long as the other formats.

The PN task was considerably more demanding, as shown by both accuracy (Figure 4) and reaction times, which were approximately twice as long (Figure 5). Some of the additional time could have been the result of choosing the numbers to type in, or the action of typing into two separate cells (fractions condition). Nevertheless, the linear model was better than the logarithmic model for both adults and children for integers, money, and decimals, but not fractions.

These results shed new light on the understanding the real value of rational numbers. It has been previously observed in number comparison tasks that when processing fractions (Bonato et al., 2007; Ni & Zhou, 2005) and decimals (Rittle-Johnson et al., 2001), both adults and children are subject to a "whole-number bias" (Ni & Zhou, 2005) where they treat fractions in a componential manner. By using number line tasks and stimulus design that does not encourage componential strategies, we show that, at least when they are given a numerical stimulus to map onto a defined physical space, both adults and children can access and correctly represent (i.e., linearly) the real value of fractions and decimals and process them in a holistic way. However, when a spatial cue has to be translated into a numerical value, fractional but not decimal stimuli require greater effort, and the mapping is not linear for either of the groups. This is compatible with the idea that when allowed or encouraged by the task, the use of componential strategies might be preferred to the access of the numerical value of the fraction (Kallai & Tzelgov, 2009).

Furthermore, the different results obtained using the two different approaches to the investigation of the representation and processing of rational numbers (i.e., number comparison paradigms and number line tasks) could be interpreted using a more general theoretical framework, which contrasts two generic modes of cognitive functions (Kahneman, 2003)—an intuitive mode in which judgements and decisions are made automatically and rapidly (which could be the process associated with number comparison tasks), and a control mode, which is deliberate and slower (to be linked with the number line estimation tasks used here).

In conclusion, the timed number line tasks used here offer a precise method for assessing number understanding. They demonstrate that fractions, but not decimals, are more difficult to mentally manipulate than other number notations, as shown by the accuracy and reaction time data. Yet, we show for the first time that, when participants are asked to place a numerical value on the number line, a linear representation is evident already in children by the age of 10 for both fractional and decimal stimuli. The same is not entirely true for the position to number task, where fractional stimuli do not yield linear responses and never seem to make the transition from exponential to a linear representation even with experience.

Finally, the current design of these tasks offers a novel and simple approach for assessing another aspect of number understanding in children and adults who are suspected of suffering from arithmetical learning disabilities, such as developmental dyscalculia.

Original manuscript received 27 August 2010
Accepted revision received 21 June 2011
First published online 21 September 2011

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